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ANOMALIES IN SPACES OF EVEN AND ODD DIMENSIONS IN THE SCHEME OF STOCHASTIC QUANTIZATION

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Axial anomalies in spaces of even and odd dimensions are studied by the method of stochastic quantization with a generalized scheme of stochastic regularization. Although the stochastic regularization formally preserves the axial and gauge symmetries, the standard even-dimensional axial anomalies of Dirac fermions are correctly reproduced in the limit in which the regularization is lifted, while the anomalies of the chiral fermions are reproduced in covariant form. An analysis is also made of the general conditions for the existence and canceling of anomalies of massless fermions that violate parity in odd-dimensional spaces. The stochastic scheme works in odd-dimensional spaces when the parity-violating anomalies are absent. The P-anomalous part of the effective action of infinitely heavy fermions in odd-dimensional spaces is calculated explicitly.

1. Introduction

The scheme of stochastic quantization [1] is attractive because of its following important properties: a) Faddeev–Popov ghosts are absent [1,2]; b) the $N \to \infty$ limit of Yang–Mills theory with SU(N) symmetry can be readily studied [3]; c) nonholonomic systems can be readily quantized [4]; d) stochastic quantization is convenient for nonperturbative Monte Carlo calculations on lattices [5].

Recently, stochastic quantization has also been applied to gauge theories with fermions [6-9] and supergauge [7-9] and gravitational theories [10]. Stochastic quantization gives an interesting possibility of a new regularization of field theories outside the framework of perturbation theory by means of a smoothing of the distribution of the stochastic currents with respect to the stochastic time, a procedure that at the first glance should not violate the symmetry properties of the original theory.

In this paper we derive the Adler–Bardeen–Jackiw anomalies [11] in the framework of stochastic quantization, and also the anomalies of chiral (left) fermions in spaces of even dimension D (in the present paper, the space is assumed to be Euclidean). These anomalies are expressed in terms of integrals of the kernel of the quadratic Dirac operator $\check{\nabla}^2(A)$ in the background field $A_\mu(x)$ with respect to the stochastic time. The final results for the anomalies are obtained after lifting of the stochastic regularization, this reducing to analysis of the behavior of the kernel at short times. This approach is similar to the method of deriving the anomalies proposed in [1]. Stochastic quantization gives the correct anomaly for the divergence of the axial current of the Dirac

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fermions. The anomalies of the fermions in the scheme of stochastic quantization are reproduced in the covariant form (which does not satisfy the Wess–Zumino conditions of integrability [13]). The connection between the two forms of the anomalies, the covariant and the self-consistent (which does satisfy the Wess–Zumino conditions of integrability), is clarified in [14].

We then consider the parity violating anomalies of massless fermions for odd dimensions D [15-22].

The interest in these anomalies is due, first, to the fact that they are analogs of the axial anomalies for odd D and, second, on account of their connection with the factorization of the fermion number [17] and the quantum Hall effect [20]. Our analysis shows that the scheme of stochastic quantization does not reproduce the parity violating anomalies. The reason is that stochastic regularization leads to a gauge-invariant and P-even expression for the fermion current, and this does not agree with the more physical expression for the current obtained from the gauge-invariant definition [18-19]* $\ln\det(-i\check{\nabla}(A))$ in terms of the η invariant of the operator $\nabla(A)$ [23,24]. In the paper, we find general conditions (on the gauge group and number of flavors) for the canceling or presence of parity violating anomalies for odd D.

We assume the standard boundary conditions for the gauge field $A_\mu(x)$:

$$\begin{gathered} A_\mu(x)=-ig^{-1}(\hat{x})(\partial_\mu g(\hat{x}))+O(|x|^{-1-\varepsilon}),\quad |x|\to\infty,\\ x=(x^0,x^1,\dots,x^{D-1})\in R^D,\quad \hat{x}=\frac{x}{|x|}\in S^{D-1},\quad g:S^{D-1}_\infty\to G=U(n). \end{gathered} \tag{1.1}$$

If the conditions (1.1) are not satisfied (for example, A_μ is a static (x^0-independent) field or $F_{\mu\nu}$ is a constant (x-independent) quantity), then for odd D an additional parity violating anomaly can arise. The criterion for its existence is the condition [25,26]

$$P_{\check{\Delta}(A)}(0;x,x)\neq 0\quad(<\infty), \tag{1.2}$$

where $P_{\check{\nabla}(A)}(\lambda;x,x')$ is the kernel of the spectral density of the Dirac operator $\check{\nabla}$:

$$\check{\nabla}(A)=\int\lambda P_{\check{\nabla}(A)}(\lambda)\,d\lambda.$$

2. Stochastic Quantization of Fermions

The basic tool of stochastic quantization — the Langevin equation for the Dirac fermions $\psi(\tau, x)$ in the background field $A_\mu(x)$ — has the form

$$\begin{gathered} \partial_\tau\psi(\tau,x)=-[\check{\nabla}^2(A)+m^2]\psi(\tau,x)+\eta(\tau,x),\quad \partial_\tau\bar{\psi}(\tau,x)=-[\check{\nabla}^2(A)+m^2]^T\bar{\psi}(\tau,x)+\bar{\eta}(\tau,x),\\ \check{\nabla}(A)=\gamma_\mu\nabla_\mu=\gamma_\mu(\partial_\mu+iA_\mu),\quad A_\mu(x)=T^aA_\mu^a(x),\quad \mu=0,\dots,D-1;\quad a=0,\dots,n^2-1;\\ \{\gamma_\mu\gamma_\nu\}=-2\delta_{\mu\nu},\quad \gamma_\mu^+=-\gamma_\mu,\quad \gamma^{(D+1)}=i^{D(D+1)/2}\gamma_0\dots\gamma_{D-1}=(\gamma^{(D+1)})^+,\\ \gamma^{(D+1)}=\pm1\ (\text{for odd } D);\quad \mathrm{tr}(T^aT^b)=n\delta^{ab},\quad T^aT^b=\delta^{ab}+(d^{abc}+if^{abc})T^c. \end{gathered} \tag{2.1}$$

In (2.1), T^a are the Hermitian generators of the group U(n), T^0 belongs to the subgroup U(1), the superscript T denotes the transpose; and the symbol + Hermitian conjugation; η, $\bar{\eta}$ are the stochastic currents with the distribution

$$\exp\left\{-\frac{1}{2}\int d\tau\,d^Dx\,\bar{\eta}(\tau,x)[(m-i\check{\nabla}(A))^{-1}\eta](\tau,x)\right\}, \tag{2.2}$$

this leading to the correlation function

$$\langle\eta(\tau,x)\bar{\eta}(\tau',x')\rangle=2\delta(\tau-\tau')[m-i\check{\nabla}(A)]\delta^{(D)}(x-x'). \tag{2.3}$$

The main assertion of the scheme of stochastic quantization is expressed by the equation

$$\langle F(\psi(x),\bar{\psi}(x))\rangle_Q=\lim_{\tau\to\infty}\langle F(\psi(\tau,x),\bar{\psi}(\tau,x))\rangle_\eta, \tag{2.4}$$

where the index Q denotes the ordinary quantum mean of the functional $F(\psi(x), \bar{\psi}(x))$, the index η denotes the stochastic mean with weight (2.2), and $\psi(\tau, x)$, $\bar{\psi}(\tau, x)$ on the

*In addition, there is a communication of A. M. Polyakov.

right-hand side of (2.4) solve Eq. (2.1) with arbitrary initial data. We take zero-value initial data at $\tau = -\infty$. Then the solutions of Eq. (2.1) take the form

$$\psi(\tau, x)=(G\eta)(\tau, x), \quad \bar{\psi}(\tau, x)=(G^T\bar{\eta})(\tau, x),$$
$$G(\tau, x; \tau', x')=\theta(\tau-\tau')\exp\{-(\tau-\tau')[m^2+\check{\nabla}^2(A)]\}(x, x'). \tag{2.5}$$

For our choice of the initial data, the passage to the limit $\tau \to \infty$ in (2.4) is not essential (in this case, (2.4) is valid for all finite τ). The mass term in Eqs. (2.1)-(2.3) is needed to achieve the equilibrium state (2.4) when the operator $\check{\nabla}(A)$ has zero modes.

The stochastic regularization [7,27] is introduced in (2.3) as follows:

$$\langle\eta(\tau, x), \bar{\eta}(\tau', x')\rangle=2\delta_\Lambda(\tau-\tau')[m-i\check{\nabla}(A)]\delta^{(D)}(x-x'),$$
$$\lim_{\Lambda\to\infty}\delta_\Lambda(\tau)=\delta(\tau), \quad \delta_\Lambda(-\tau)=\delta_\Lambda(\tau), \quad \partial_\tau^k\delta_\Lambda(\tau)|_{\tau=0}=0; \quad k=0,\dots,L-1, \tag{2.6}$$

where L is an integer. The function $\delta_\Lambda(\tau)$ can be realized, for example, as follows: $\delta_\Lambda(\tau)={}^1/_2(L!)^{-1}\Lambda(\Lambda|\tau|)^L\exp(-\Lambda|\tau|)$.

We also formulate the scheme of stochastic quantization for chiral (left) fermions in the background field $\psi_L(\tau, x)$ for even D. In this case, Eqs. (2.1), (2.3), and (2.6) are replaced by

$$\partial_\tau\psi_L{}^\perp=-(\mathscr{D}\mathscr{D}^+)\psi_L{}^\perp+\eta_L{}^\perp, \quad \partial_\tau\bar{\psi}_L{}^\perp=-(\mathscr{D}^+\mathscr{D})^T\bar{\psi}_L{}^\perp+\bar{\eta}_L{}^\perp, \tag{2.7}$$

$$\langle\eta_L{}^\perp(\tau, x)\bar{\eta}_L{}^\perp(\tau', x')\rangle=-2i\delta_\Lambda(\tau-\tau')\mathscr{D}(1-\Pi_0)\delta^{(D)}(x-x'), \tag{2.8}$$

where we have used the notation

$$\gamma_\mu=i\begin{pmatrix}0 & \sigma_\mu\\ \sigma_\mu{}^+ & 0\end{pmatrix}, \quad \sigma_\mu\sigma_\nu{}^++\sigma_\nu\sigma_\mu{}^+=\delta_{\mu\nu}, \quad \sigma_\mu{}^+\sigma_\nu+\sigma_\nu{}^+\sigma_\mu=\delta_{\mu\nu},$$

$$\check{\nabla}(A)=\begin{pmatrix}0 & \mathscr{D}\\ \mathscr{D}^+ & 0\end{pmatrix}, \quad \check{\nabla}^2(A)=\begin{pmatrix}\mathscr{D}\mathscr{D}^+ & 0\\ 0 & \mathscr{D}^+\mathscr{D}\end{pmatrix}, \quad \gamma^{(D+1)}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.9}$$
$$\mathscr{D}^+=i\sigma_\mu\nabla_\mu(A), \quad \psi_L{}^\perp\equiv(1-\bar{\Pi}_0)\psi_L, \quad \bar{\psi}_L{}^\perp\equiv(1-\Pi_0)\bar{\psi}_L,$$

and similarly for η.

In (2.8) and (2.9), Π_0 and $\bar{\Pi}_0$ are the projection operators of the zero modes of the operators $\mathscr{D}^+\mathscr{D}$ and $\mathscr{D}\mathscr{D}^+$, respectively. In the calculation of the stochastic mean values (2.4) it is helpful to use the equations

$$\bar{\Pi}_0\mathscr{D}=\mathscr{D}\Pi_0, \quad \mathscr{D}^+\bar{\Pi}_0=\Pi_0\mathscr{D}^+, \quad \mathscr{D}\exp\{-\tau\mathscr{D}^+\mathscr{D}\}=(\exp\{-\tau\mathscr{D}\mathscr{D}^+\})\mathscr{D},$$
$$\mathscr{D}^+\exp\{-\tau\mathscr{D}\mathscr{D}^+\}=(\exp\{-\tau\mathscr{D}^+\mathscr{D}\})\mathscr{D}^+. \tag{2.10}$$

We have in fact eliminated from the Langevin equations the zero modes, and therefore the equilibrium state (as in (2.4)) can be attained.

Note that (2.8) explicitly preserves the chiral gauge invariance.

When zero-value boundary conditions at $\tau = -\infty$ are chosen, the solutions of Eq. (2.7) have the form

$$\psi_L{}^\perp(\tau, x)=(G_L\eta_L{}^\perp)(\tau, x), \quad \bar{\psi}_L{}^\perp(\tau, x)=(\bar{G}_L{}^T\bar{\eta}_L{}^\perp)(\tau, x);$$
$$G_L(\tau, x; \tau', x')=\theta(\tau-\tau')\exp\{-(\tau-\tau')\mathscr{D}\mathscr{D}^+\}(x, x'), \tag{2.11}$$
$$\bar{G}_L(\tau, x; \tau', x')=\theta(\tau-\tau')\exp\{-(\tau-\tau')\mathscr{D}^+\mathscr{D}\}(x, x').$$

3. Chiral Anomalies in Stochastic Quantization

We now turn to the discussion of chiral anomalies for even D in the scheme of stochastic quantization. We first consider the covariant divergence of the axial current of the Dirac fermions:

$$\nabla_\mu{}^{ab}J_\mu^{(D+1)b}(x)\equiv\nabla_\mu{}^{ab}\langle\bar{\psi}T^b(-i\gamma_\mu)\gamma^{(D+1)}\psi\rangle_\eta=$$
$$-2m\langle\bar{\psi}T^a\gamma^{(D+1)}\psi\rangle_\eta+\langle[(m-i\check{\nabla}(A)^T)^{-1}(\bar{\eta}-\partial_\tau\psi)]T^a\gamma^{(D+1)}\psi\rangle_\eta+$$

$$\langle\bar{\psi}T^a\gamma^{(D+1)}[(m-i\check{\nabla}(A))^{-1}(\eta-\partial_\tau\psi)]\rangle_\eta, \tag{3.1}$$

where $\nabla_\mu^{ab}=\delta^{ab}\partial_\mu+f^{abc}A_\mu^c(x)$. Substituting (2.5) in the right-hand side of (3.1), averaging with respect to the stochastic source in accordance with (2.6), and using the operator relation

$$\int_{\tau_1}^{\tau_2} d\tau\exp(-\tau H)=H^{-1}\{\exp(-\tau_1 H)-\exp(-\tau_2 H)\},$$

we obtain

$$\nabla_\mu^{ab}J_\mu^{(D+1)b}(x)=2m^2\int_0^\infty d\tau(\exp(-\tau m^2))\operatorname{tr}[T^a\gamma^{(D+1)}(\exp(-\tau\check{\nabla}^2(A)))(x,x)]\int_{-\tau}^{\tau}d\tau'\delta_\Lambda(\tau')-$$

$$4\int_0^\infty d\tau\delta_\Lambda(\tau)(\exp(-\tau m^2))\operatorname{tr}[T^a\gamma^{(D+1)}(\exp(-\tau\check{\nabla}^2(A)))(x,x)], \tag{3.2}$$

where the first term on the right-hand side arises from the term $-2m\langle\bar{\psi}T^a\gamma^{(D+1)}\psi\rangle_\eta$.

The presence of $\delta_\Lambda(\tau)$ in (3.2) regularizes the ultraviolet divergences manifested in the form of singularities of the type $O(\tau^{-k})$, $k \geqq 1$, in the integral over τ. These divergences can be analyzed explicitly by means of Seeley's expansion [28]

$$(\exp\{-\tau\check{\nabla}^2(A)\})(x,x)=\sum_{j=0}^\infty \tau^{(j-D)/2}\Phi_{(j-D)/2}^{(D)}(\check{\nabla}^2(A);x), \tag{3.3}$$

where $\Phi_{(j-D)/2}^{(D)}$ are local gauge-invariant functionals of dimension j. A recursive procedure for calculating these functionals correctly is described in Appendix A.

Using (A.3), (A.5), and (A.9), we can readily show that

$$\operatorname{tr}[T^a\gamma^{(D+1)}\Phi_{(j-D)/2}^{(D)}(\check{\nabla}^2(A);x)]=0 \quad \text{for} \quad j<D.$$

Thus, when we go to the limit $\Lambda\to\infty$ in (3.2) there are no ultraviolet divergences,

$$\nabla_\mu^{ab}J_\mu^{(D+1)b}(x)=2\int_0^\infty d\alpha(\exp(-\alpha))\operatorname{tr}[T^a\gamma^{(D+1)}\times$$

$$\exp(-((\alpha/m^2)\check{\nabla}^2(A)))(x,x)]-2\operatorname{tr}[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\check{\nabla}^2(A);x)]. \tag{3.4}$$

The second term on the right-hand side of (3.4) — it does not depend on the mass m — can be readily transformed (by means of (A.3), (A.5), and (A.9)) to

$$-2\operatorname{tr}[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\check{\nabla}^2(A);x)]=-2[(D/2)!(4\pi)^{D/2}]^{-1}\varepsilon_{\mu_1\ldots\mu_D}\operatorname{tr}[T^aF_{\mu_1\mu_2}\ldots F_{\mu_{D-1}\mu_D}] \tag{3.5}$$

(and this is equal to $-2C_{D/2}(F;x)$ for a = 0), where $C_{D/2}(F;x)$ is the density of D/2-th characteristic Chern class (see, for example, [29]). For the first term on the right-hand side of (3.4) in the limit $m\to\infty$ we find, using (B.2a) in the Appendix,

$$2\operatorname{tr}[T^a\gamma^{(D+1)}\Pi_0^{\check{\nabla}(A)}(x,x)] \tag{3.6}$$

(and this is equal to 2 index $(\check{\nabla}(A);x)$ for a = 0), where $\Pi_0^{\check{\nabla}(A)}$ denotes the kernel of the projection operator of the zero modes of the operator $\check{\nabla}(A)$, and index$(\check{\nabla};x)$ denotes the density of the index of the operator $\check{\nabla}(A)$ (see [12]). Collecting together (3.5) and (3.6), we obtain the standard form of the D-dimensional Abelian [12,30] and non-Abelian [31] anomalies:

$$\partial^\mu J_\mu^{(D+1)a=0}(x)=2[\operatorname{index}(\check{\nabla}(A);x)-C_{D/2}(F;x)], \tag{3.7}$$

$$\nabla_\mu^{ab}J_\mu^{(D+1)b}(x)=2\{\operatorname{tr}[T^a\gamma^{(D+1)}\Pi_0^{\check{\nabla}(A)}(x,x)]-[(D/2)!(4\pi)^{D/2}]^{-1}\varepsilon_{\mu_1\ldots\mu_D}\operatorname{tr}[T^aF_{\mu_1\mu_2}\ldots F_{\mu_{D-1}\mu_D}]\}, \tag{3.8}$$

where a, b = 1, ..., n^2-1.

We also briefly discuss the chiral anomalies of the left fermions in stochastic quantization. Using Eqs. (2.7), (2.8), (2.11), (2.10), and (3.3), we find

$$\nabla_\mu^{ab}J_\mu^{L,b}(x)\equiv\nabla_\mu^{ab}\langle\bar{\psi}_L{}^\perp(\tau,x)\sigma_\mu{}^+\psi_L{}^\perp(\tau,x)\rangle_\eta=\lim_{\Lambda\to\infty}2\int_0^\infty d\tau\delta_\Lambda(\tau)\operatorname{tr}\{T^a[(\exp(-\tau\mathscr{D}^+\mathscr{D}))\times$$

$$(1-\Pi_0)(x,x)-(\exp(-\tau\mathscr{D}\mathscr{D}^+))(1-\bar{\Pi}_0)(x,x)]\}=$$

$$\mathrm{tr}[T^a(\bar{\Pi}_0(x,x)-\Pi_0(x,x))]+\mathrm{tr}[T^a(\Phi_0^{(D)}(\mathscr{D}^+\mathscr{D};x)-\Phi_0^{(D)}(\mathscr{D}\mathscr{D}^+;x))], \tag{3.9}$$

where Φ_0 denotes Seeley's zeroth coefficient, which can be readily calculated in accordance with the algorithm described in Appendix A. Using Eq. (3.5) and the representation $\gamma^{(D+1)}$ (2.9), Eq. (3.9) can be transformed as follows:

$$\nabla_\mu^{ab}J_\mu^{L,b}(x)=\mathrm{tr}[T^a\gamma^{(D+1)}\Pi_0^{\check{\nabla}(A)}(x,x)]-\mathrm{tr}[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\check{\nabla}^2(A);x)]={}^1/_2\nabla_\mu^{ab}J_\mu^{(D+1)b}(x), \tag{3.10}$$

where the last term is given by Eq. (3.7) when a = 0 and by Eq. (3.8) for a = 1, ..., $n^2 - 1$.

Equation (3.10) shows that the anomalies of the chiral fermions of the scheme of stochastic quantization are reproduced in the covariant form. The connection between the covariant and self-consistent forms of the anomalies is discussed in [14].

4. Parity Violating Anomalies and Problems of Stochastic Quantization in Odd-Dimensional Spaces

We consider the fermion current in stochastic quantization:

$$J_\mu^a(x)\equiv\langle\bar{\psi}T^a(-i\gamma_\mu)\psi\rangle_\eta=\int_0^\infty d\tau\exp(-\tau m^2)\left(\int_{-\tau}^{\tau}d\tau'\delta_\Lambda(\tau')\right)\times$$

$$\mathrm{tr}\{T^a\gamma_\mu[(\check{\nabla}(A)+im)\exp(-\tau\check{\nabla}^2(A))](x,x)\}. \tag{4.1}$$

We apply Seeley's expansion (3.3) to Eq. (4.1):

$$\{\check{\nabla}(A)\exp(-\tau\check{\nabla}^2(A))\}(x,x)=\sum_{j=0}^{\infty}\tau^{(j-D-1)/2}\,\Phi_{(j-D-1)/2}^{(D)}(\check{\nabla}^2(A)\,|\,\check{\nabla}(A))(x). \tag{4.2}$$

For odd D, we obtain by means of (A.3), (A.5), (A.6), and (A.9)

$$\mathrm{tr}[T^a\gamma_\mu\Phi_{(j-D)/2}^{(D)}(\check{\nabla}^2(A)\,|\,\check{\nabla}(A))(x)]=\mathrm{tr}[T^a\gamma_\mu\Phi_{(j-D)/2}^{(D)}(\check{\nabla}^2(A);x)]=0,\quad j\leqslant D-2. \tag{4.3}$$

We see that the right-hand side of (4.1) does not have ultraviolet divergences, i.e., singularities as $\tau \to 0$, when the regularization is lifted. The final result has the form

$$J_\mu^a(x)=\int_0^\infty d\tau\exp(-\tau m^2)\mathrm{tr}[T^a\gamma_\mu(\check{\nabla}(A)\exp(-\tau\check{\nabla}^2(A)))(x,x)]+im\int_0^\infty d\tau\exp(-\tau m^2)\times$$

$$\mathrm{tr}[T^a\gamma_\mu\exp(-\tau\check{\nabla}^2(A))(x,x)]. \tag{4.4}$$

Under P transformations (D odd) we have

$$\psi\xrightarrow{P}\psi(\tau,x\,|\,A_\mu^P,\eta^P,m^P)=-i\gamma_1\psi(\tau,x^P\,|\,A_\mu,\eta,m), \tag{4.5}$$

where in the argument of the solution $\psi(\tau, x)$ of Eq. (2.1) we have explicitly indicated the dependence of the solution on $A_\mu(x)$, $\eta(\tau, x)$, and m and we have used the notation

$$A_\mu^P(x)=(A_0,-A_1,A_2,\dots,A_{D-1})(x^P),\quad \eta^P(\tau,x)=-i\gamma_1\eta(\tau,x^P),$$
$$m^P=-m,\quad x^P\equiv(x^0,-x^1,x^2,\dots,x^{D-1}). \tag{4.6}$$

The kernel of the Dirac operator is transformed as follows:

$$\check{\nabla}(A^P)(x,x')=\gamma_1\check{\nabla}(A)(x^P,x'^P)\gamma_1. \tag{4.7}$$

In accordance with (4.7), the first term on the right-hand side of (4.4) has normal parity, and it can be rewritten as

$$J_\mu^{a\,(\mathrm{norm})}(x)\equiv\int_0^\infty d\tau\exp(-\tau m^2)\mathrm{tr}[T^a\gamma_\mu(\check{\nabla}(A)\times$$

$$\exp(-\tau\check{\nabla}^2(A)))(x,x)]=(2i)^{-1}\frac{\delta}{\delta A_\mu^a(x)}\ln\det[m^2+\check{\nabla}^2(A)]. \tag{4.8}$$

The second term on the right-hand side of (4.4) can lead to parity violation if it does not vanish in the limit $m \to 0$:

$$J_\mu^{a(\mathrm{PV})}(x) \equiv im \int_0^\infty d\tau \exp(-\tau m^2) \operatorname{tr}[T^a \gamma_\mu \times$$

$$(\exp(-\tau \check{\nabla}^2(A)))(x,x)] = \frac{\delta}{\delta A_\mu^a(x)} \left\{ \left(-\frac{m}{2}\right) \int_0^\infty d\tau \left(\frac{\pi}{\tau}\right)^{1/2} \times \right.$$

$$\exp(-\tau m^2) \operatorname{Tr}[\operatorname{Erfc}(\tau^{1/2} \check{\nabla}(A)) - \operatorname{Erfc}(\tau^{1/2} \check{\partial})], \quad \operatorname{Erfc}(\alpha) = 2\pi^{-1/2} \int_\alpha^\infty d\beta \exp(-\beta^2), \tag{4.9}$$

$$\lim_{m \to 0} J_\mu^{a(\mathrm{PV})}(x) = \lim_{m \to 0} im^{-1} \int_0^\infty d\alpha \exp(-\alpha) \times$$

$$\operatorname{tr}[T^a \gamma_\mu (\exp((-\alpha/m^2) \check{\nabla}^2(A)))(x,x)] = i\pi \operatorname{sign}(m) \operatorname{tr}[T^a \gamma_\mu P_{\check{\nabla}(A)}(0;x,x)]. \tag{4.10}$$

In (4.10), we have used Eq. (B.2):

$$\lim_{\tau \to \infty} \exp(-\tau \check{\nabla}^2(A))(x,x) \approx$$

$$\begin{cases} \Pi_0^{\check{\nabla}(A)}(x,x), & \text{if } \check{\nabla}(A) \text{ has zero modes,} \\ (\pi/\tau)^{1/2} P_{\check{\nabla}(A)}(0;x,x), & \text{if } \check{\nabla}(A) \text{ does not have zero modes,} \end{cases} \tag{4.11}$$

where $P_{\check{\nabla}(A)}$ is the kernel of the spectral plane $\check{\nabla}(A)$.

The parity violation in (4.10) must be interpreted as spontaneous (not anomalous) since the appearance of the factor sign(m) in Eq. (4.10) shows that the ground state is degenerate when the parity violating mass term $m\bar{\psi}\psi$ (see (4.5)) is ignored.

As is shown in Appendix B, for the standard boundary conditions (1.1)

$$P_{\check{\nabla}(A)}(0;x,x) = \begin{cases} \delta(0)\Pi_0^{\check{\nabla}(A)}(x,x) = \infty, & \text{if } \check{\nabla}(A) \text{ has zero modes,} \\ 0, & \text{if } \check{\nabla}(A) \text{ does not have zero modes.} \end{cases} \tag{4.12}$$

Therefore we obtain (up to ordinary infrared divergences associated with the zero modes of the operator $\check{\nabla}(A) J_\mu^{a(\mathrm{PV})}(x)|_{m=0} = 0$) a P-invariant result for the total current as well in the scheme of stochastic quantization:

$$J_\mu^{a\,(\mathrm{SQ})}(x)|_{m=0} = \lim_{m \to 0} J_\mu^{a(\mathrm{norm})}(x) = (2i)^{-1} \frac{\delta}{\delta A_\mu^a(x)} \ln\det \check{\nabla}^2(A). \tag{4.13}$$

However, the result (4.4), (4.13) is not identical to the well-known expression

$$\mathcal{J}_\mu^a(x) = \langle \bar{\psi}(x) T^a(-i\gamma_\mu)\psi(x) \rangle_Q = -i \frac{\delta}{\delta A_\mu^a(x)} \ln\det[-(m + i\check{\nabla}(A))], \tag{4.14}$$

which presupposes the gauge-invariant definition of $\ln\det[-(m + i\check{\nabla}(A))]$ [18,19]*:

$$\ln\det[-(m + i\check{\nabla}(A))] = {}^1/_2 \ln\det(m^2 + \check{\nabla}^2(A)) -$$

$$i\frac{\pi}{2}\eta_{\check{\nabla}}(A) - \frac{im}{2}\int_0^\infty d\tau \left(\frac{\pi}{\tau}\right)^{1/2} \exp(-\tau m^2) \times$$

$$\operatorname{Tr}[\widetilde{\operatorname{Erfc}}(\tau^{1/2}\check{\nabla}(A)) - \widetilde{\operatorname{Erfc}}(\tau^{1/2}\check{\partial}))] - S_{\mathrm{ct}}(A), \quad \widetilde{\operatorname{Erfc}}(\alpha) = \operatorname{sign}(\alpha) \operatorname{Erfc}(|\alpha|); \tag{4.15}$$

$$\ln\det(m^2 + \check{\nabla}^2(A)) = -\int_0^\infty d\tau \tau^{-1} \exp(-\tau m^2) \times$$

$$\operatorname{Tr}[\exp(-\tau\check{\nabla}^2(A)) - \exp(-\tau\partial^2)] = -\frac{d}{ds}\zeta_{[m^2 + \check{\nabla}^2(A)]}(s)|_{s=0}; \tag{4.16}$$

*See the footnote on page 1275.

$$\eta_{\check{\nabla}}(A) = \int_{-\infty}^{+\infty} d\lambda \, \mathrm{sign}\, \lambda \, \mathrm{Tr}\, [P_{\check{\nabla}(A)}(\lambda) - P_{\partial}(\lambda)] =$$

$$\int_0^\infty d\tau \, (\pi\tau)^{-1/2} \, \mathrm{Tr}\, [\check{\nabla}(A) \exp(-\tau\check{\nabla}^2(A)) - \partial \exp(-\tau\partial^2)]. \tag{4.17}$$

By means of Seeley's expansion (3.3), (4.2) and (A.3), (A.5), (A.6), (A.8), and (A.10) we can readily show that for odd D there are no ultraviolet divergences in the integrals (4.16) and (4.17) over τ. The operator traces in (4.15)-(4.17) are also well defined by virtue of the existence and completeness of the wave operators $U_{\pm}(\check{\nabla}^2(A), -\partial^2)$ (in the sense of scattering theory) for the boundary conditions (1.1) (see Appendix B). The second equation in (4.16) is the ζ regularization [32] of ln det of the positive operator $\check{\nabla}^2 + m^2$, and $\eta_{\check{\nabla}}(A)$ (4.17) is the η invariant [23] of the operator $\check{\nabla}(A)$. Actually, the η regularization of the function ln det (4.15) is formally correct for all self-adjoint elliptical operators. For $\eta_{\check{\nabla}}(A)$ in (4.17) we have the following properties with respect to gauge and P transformations:

$$\eta_{\check{\nabla}}(A^g) = \eta_{\check{\nabla}}(A), \quad \eta_{\check{\nabla}}(A^P) = -\eta_{\check{\nabla}}(A). \tag{4.18}$$

The appearance of $\eta_{\check{\nabla}}$ in (4.15) represents in accordance with (4.18) the general form of the parity violating anomaly (as $m \to 0$) if the anomaly is not cancelled by a suitable choice of the counterterm $S_{ct}(A)$ in Eq. (4.15). We now discuss the problem of the canceling of the parity violating anomaly by means of a local counterterm $S_{ct}(A)$.

Using the operator identity

$$\delta\{\mathrm{Tr}\,[Q \exp(-\tau Q^2)]\} = (d/d\tau^{1/2})\{\tau^{1/2}\, \mathrm{Tr}\, \delta Q \exp(-\tau Q^2)\}$$

and (A.1) and (A.2), we obtain from Eq. (4.17)

$$(\delta/\delta A_\mu{}^a(x))\, \eta_{\check{\nabla}}(A) = -2i\pi^{-1/2}\, \mathrm{tr}\, [T^a \gamma_\mu \Phi^{(D)}_{-1/2}(\check{\nabla}^2(A); x)] + 2i\, \mathrm{tr}\, [T^a \gamma_\mu P_{\check{\nabla}(A)}(0; x, x)]. \tag{4.19}$$

Direct calculation of the first term on the right-hand side of (4.19) by means of (A.3), (A.6), and (A.10) gives

$$(-1)^{(D+1)/2}\pi^{-1}\left[\left(\frac{D-1}{2}\right)!\;(4\pi)^{(D-1)/2}\right]^{-1} \varepsilon_{\mu\mu_1\ldots\mu_{D-1}} \mathrm{tr}[T^a F_{\mu_1\mu_2}\ldots F_{\mu_{D-2}\mu_{D-1}}] =$$

$$(-1)^{(D+1)/2} 2 (\delta/\delta A_\mu{}^a(x))\, W^{(D)}_{\mathrm{Ch\text{-}S}}(A), \tag{4.20}$$

where $W^{(D)}_{\mathrm{Ch\text{-}S}}(A)$ is the well-known Chern–Simons term [29]:

$$W^{(D)}_{\mathrm{Ch\text{-}S}}(A^g) = W^{(D)}_{\mathrm{Ch\text{-}S}}(A) + N_D(g), \qquad W^{(D)}_{\mathrm{Ch\text{-}S}}(A^P) = -W^{(D)}_{\mathrm{Ch\text{-}S}}(A); \tag{4.21}$$

$$N_D(g) = -\left(\frac{i}{2\pi}\right)^{(D+1)/2}\left(\frac{D-1}{2}\right)!\;(D!)^{-1}\varepsilon_{\mu_1\ldots\mu_D}\int d^D x\, \mathrm{tr}[(g^{-1}\partial_{\mu_1} g)\ldots(g^{-1}\partial_{\mu_D} g)]. \tag{4.22}$$

The topological charge $N_D(g)$ (4.22) has the properties [33]

$$N_D(g) = \begin{cases} \mathbb{Z}, & \text{if } \pi_D(U(n)) = \mathbb{Z}, \text{ i.e., for odd } D < 2n, \\ 0, & \text{if } \pi_D(U(n)) \neq \mathbb{Z}, \text{ i.e., for odd } D > 2n. \end{cases} \tag{4.23}$$

For example, for D = 3 the Chern–Simons term has the form

$$W^{(3)}_{\mathrm{Ch\text{-}S}} = (16\pi^2)^{-1}\varepsilon_{\mu\nu\lambda}\int d^3x\, \mathrm{tr}\{A_\mu F_{\nu\lambda}(A) - i^2/_3 A_\mu A_\nu A_\lambda\}, \tag{4.24}$$

and this is the well-known topological mass term of the field $A_\mu(x)$ [34].

By means of (4.19), (4.20), and (4.12) we finally obtain [23]

$$\eta_{\check{\nabla}}(A) = (-1)^{(D+1)/2} 2 W^{(D)}_{\mathrm{Ch\text{-}S}}(A) + B(A), \tag{4.25}$$

where B(A) has the properties

$$(\delta/\delta A_\mu{}^a(x))\, B(A) = 2i\, \mathrm{tr}\, [T^a \gamma_\mu P_{\check{\nabla}(A)}(0; x, x)] =$$

$$\begin{cases} \infty, & \text{if } \check{\nabla}(A) \text{ has zero modes}, \\ 0, & \text{if } \check{\nabla}(A) \text{ does not have zero modes}, \end{cases} \tag{4.26a}$$

$$B(A^g)=B(A)+(-1)^{(D-1)/2}2N_D(g), \quad B(A^P)=-B(A), \tag{4.26b}$$

$$B(A)=0, \quad \text{if} \quad \pi_D(U(n))\neq\mathbb{Z}, \text{i.e.}, D>2n \tag{4.26c}$$

(see (4.23)).

Thus, B(A) is a piecewise constant functional and takes even integral values. Indeed, B(A) can be identified with the index of a suitable (D + 1)-dimensional Dirac operator with coefficient 2 (see [23]) with even D + 1. These properties of B(A) are true for the boundary conditions (1.1), which make it possible to identify the space R^D with the compact space S^D. But if $F_{\mu\nu}(A)$ does not vanish as $|x| \to \infty$, Eq. (1.2) can be satisfied, and the term $2i\,\mathrm{tr}\,[T^a\gamma_\mu P_{\check\nabla(A)}(0;\,x,\,x)]\neq 0$ $(<\infty)$ in Eqs. (4.19) and (4.26a) will represent a nontrivial boundary effect. Since in this case the space is not compact, this boundary effect is determined by the asymptotic (scattering theory) properties of the operator $\check\nabla^2(A)$ (Appendix B).

Substituting now (4.25) in (4.15) in the limit $m \to 0$ in the case of $N_f \geqq 1$ fermion flavors, we obtain

$$N_f \ln\det[-i\check\nabla(A)]={}^1\!/_2N_f\ln\det[\check\nabla^2(A)]-{}^1\!/_2i\pi N_fB(A)+(-1)^{(D-1)/2}i\pi N_fW^{(D)}_{\text{Ch-S}}(A)-N_fS_{\text{ct}}(A). \tag{4.27}$$

There are now the following possibilities for parity violating anomalies in Eq. (4.27) (the case D = 3, see [15]):

I. If $\pi_D(U(n))=\mathbb{Z}$ (i.e., D > 2n) or $\pi_D(U(n))=\mathbb{Z}$ (i.e., D < 2n) simultaneously with the condition that N_f is even, we can choose the counterterm

$$S_{\text{ct}}(A)=i\pi(-1)^{(D-1)/2}W_{\text{Ch-S}}(A), \tag{4.28}$$

i.e.,

$$N_f\ln\det(-i\check\nabla(A))={}^1\!/_2N_f\ln\det(\check\nabla^2(A))-{}^1\!/_2i\pi N_fB(A). \tag{4.29}$$

Therefore, the parity violating anomalies (for the boundary condition (1.1)) for $A_\mu(x)$ are eliminated: $\frac{1}{2}\pi N_fB(A) = 0 \pmod{2\pi}$ under these conditions.

II. If $\pi_D(U(n))=\mathbb{Z}$ (i.e., D < 2n) and simultaneously N_f is odd, the choice (4.28) for elimination of the parity violating anomalies is not good, since we violate the gauge invariance of Eq. (4.29):

$$N_f\{\ln\det[-i\check\nabla(A^g)]-\ln\det[-i\check\nabla(A)]\}=$$
$$i\pi(-1)^{(D+1)/2}N_fN_D(g)\neq 0 \pmod{2\pi} \qquad (g(x)\in SU(n)). \tag{4.30}$$

Thus, in case II (for the standard boundary condition (1.1)), parity violating anomalies are inescapable.

The analysis of the anomalies presented above in the language of the current $J^a_\mu(x)$ (4.14) has in accordance with (4.27)-(4.30) the form

$$J_\mu{}^a(x)|_{m=0}=N_f\int_0^\infty d\tau\,\mathrm{tr}\,[T^a\gamma_\mu(\check\nabla(A)\exp(-\tau\check\nabla^2(A)))(x,x)]-i\pi N_f\,\mathrm{tr}\,[T^a\gamma_\mu P_{\check\nabla(A)}(0;x,x)]+$$
$$\begin{cases} 0 \quad \text{for condition I,} \\ \dfrac{1}{2}N_f(-1)^{(D-1)/2}\left[\left(\dfrac{D-1}{2}\right)!\,(4\pi)^{(D-1)/2}\right]^{-1}\varepsilon_{\mu\mu_1\ldots\mu_{D-1}}\times \\ \times\,\mathrm{tr}\,[T^a\hat F_{\mu_1\mu_2}\ldots\hat F_{\mu_{D-2}\mu_{D-1}}], \quad \text{for condition II,}\end{cases} \tag{4.31}$$

where $\hat F_{\mu\nu}$ denotes the non-Abelian SU(n) part of $F_{\mu\nu}(A)$.

Taking into account now the equation $P_{\check\nabla(A)}(0;\,x,\,x)=0$ for the boundary conditions (1.1) (if $\check\nabla(A)$ does not have zero modes), we can readily see from (4.31) that stochastic quantization (Eqs. (4.8) and (4.10)) reproduces the correct current (4.31) for condition I, i.e., when there are no parity violating anomalies.

For completeness, we also mention that parity may also be violated spontaneously (when there are no parity violating anomalies). This problem is discussed in [35] and [21].

5. Conclusions

In the present paper, we have determined the domain of applicability of stochastic quantization with stochastic regularization. This scheme correctly reproduces the axial anomalies of the Dirac fermions for even D. At the same time, the anomalies of the chiral fermions are reproduced in the covariant form. The scheme of stochastic quantization does not reproduce the parity violating anomalies of the massless fermions for odd D (and this occurs when the homotopy group $\pi_D(U(n))=\mathbb{Z}$ $(D<2n)$ and simultaneously the number N_f of flavors is odd). For odd D, stochastic quantization works in the absence of parity violating anomalies $\pi_D(U(n))=\mathbb{Z}$ (i.e., $D > 2n$) or ($\pi_D(U(n))\neq\mathbb{Z}$ and simultaneously N_f even).

We also make a remark concerning the criterion of spontaneous breaking of the chiral symmetry in the external field $A_\mu(x)$ for even D:

$$\lim_{m\to 0}\langle\bar{\psi}\psi\rangle_Q\neq 0 \quad (<\infty).$$

The following equation [26],

$$\lim_{m\to 0}\langle\bar{\psi}\psi\rangle_Q=-\pi\,\mathrm{sign}\,(m)\,\mathrm{tr}\,[P_{\check{\nabla}(A)}(0;x,x)], \tag{5.1}$$

shows in conjunction with (4.12) that spontaneous violation occurs under the condition of nonvanishing of $F_{\mu\nu}(A)$ as $|x|\to\infty$. In particular, in the case of a static field $A_\mu(\mathbf{x})$, substituting (B.11) in (5.1), we obtain

$$\lim_{m\to 0}\langle\bar{\psi}\psi\rangle_Q=-\pi\,\mathrm{sign}\,(m)\,\mathrm{tr}\,[\Pi_0^{(\check{\nabla}_{D-1}+iA_0)}(\mathbf{x},\mathbf{x})+\Pi_0^{(\check{\nabla}_{D-1}-iA_0)}(\mathbf{x},\mathbf{x})], \tag{5.2}$$

this being the local version of the corresponding assertion of [36]. We see that the criterion of spontaneous violation of chiral symmetry for even D is analogous to the corresponding criterion for odd D (4.10).

Our final remark concerns the limit $|m|\to\infty$ of Eq. (4.15) [21,22].

With allowance for (4.19) and (4.20) and the equation

$$\begin{aligned} N_f^{-1}J_\mu{}^a(x)&=m^{-2}\int_0^\infty d\alpha\exp(-\alpha)\;\mathrm{tr}\left[T^a\gamma_\mu\left(\check{\nabla}(A)\exp\left(-\frac{\alpha}{m^2}\check{\nabla}^2(A)\right)\right)(x,x)\right]+\\ &im^{-1}\int_0^\infty d\alpha\exp(-\alpha)\,\mathrm{tr}\left[T^0\gamma_\mu\left(\exp\left(-\frac{\alpha}{m^2}\check{\nabla}^2(A)\right)\right)(x,x)\right]+\\ &\begin{cases}0 & \text{for condition I,}\\ (-1)^{(D-1)/2}\pi\,(\delta/\delta A_\mu{}^0(x))\,W^{(D)}_{\mathrm{Ch\text{-}S}}(A) & \text{for condition II}\end{cases}\end{aligned} \tag{5.3}$$

we obtain

$$\lim_{|m|\to\infty}N_f^{-1}J_\mu{}^a(x)=\begin{cases}(-1)^{(D-1)/2}\,\mathrm{sign}\,(m)\,\pi\,(\delta/\delta A_\mu{}^a(x))\,W^{(D)}_{\mathrm{Ch\text{-}S}}(A) & \text{for condition I,}\\ (-1)^{(D-1)/2}\,[1+\mathrm{sign}\,(m)]\,\pi\,(\delta/\delta A_\mu{}^a(x))\,W^{(D)}_{\mathrm{Ch\text{-}S}} & \text{for condition II.}\end{cases} \tag{5.4}$$

From Eq. (5.4) there now follows a P-anomalous effective action for odd D:

$$\lim_{|m|\to\infty}N_f\ln\det[-(m+i\check{\nabla}(A))]=i\pi\,\mathrm{sign}\,(m)\,(-1)^{(D-1)/2}N_fW^{(D)}_{\mathrm{Ch\text{-}S}}(A)\ \text{for condition I,} \tag{5.5a}$$

$$\lim_{|m|\to\infty}N_f\ln\det[-(m+i\check{\nabla}(A))]= i\pi\,[1+\mathrm{sign}\,(m)]\,(-1)^{(D-1)/2}N_fW^{(D)}_{\mathrm{Ch\text{-}S}}(A)\ \text{for condition II.} \tag{5.5b}$$

Note that in (5.5b) the parity violating anomaly due to the η regularization (4.15) and the anomaly that arises in the limit $m\to-\infty$ cancel each other.

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Appendix A: Behavior of the Kernel of the Operator $\check{\nabla}^2(A)$ at Short Times

We consider Seeley's expansion [28] of the kernel $(\exp(-\tau\check{\nabla}^2(A)))(x,x)$ at short τ ($A_\mu(x)$ satisfies the boundary condition (1.1)):

$$(\exp(-\tau\check{\nabla}^2(A)))(x,x)=\sum_{j=0}^{\infty}\tau^{(j-D)/2}\Phi^{(D)}_{(j-D)/2}(\check{\nabla}^2(A);x). \tag{A.1}$$

The Seeley coefficients $\Phi^{(D)}_{(j-D)/2}$ are gauge-invariant functionals of dimension j and are determined by means of the symbol $\sigma(x;\ \xi,\ \lambda)$ of the operator $\check{\nabla}^2(A) - \lambda$:

$$[\check{\nabla}^2(A)-\lambda]\delta^{(D)}(x-x')=(2\pi)^{-D}\int d^D\xi\,\sigma(x;\xi,\lambda)\exp i\xi(x,x'),$$

$$\sigma(x;\xi,\lambda)=\sum_{k=0}^{2}\sigma_k(x;\xi,\lambda),\quad \sigma_k(x;\rho\xi,\rho^2\lambda)=\rho^k\sigma_k(x;\xi,\lambda)\quad(\rho>0), \tag{A.2}$$

$$\sigma_2(x;\xi,\lambda)=\xi^2-\lambda,\quad \sigma_1(x;\xi,\lambda)=2\xi_\mu A_\mu(x),\quad \sigma_0(x,\xi,\lambda)=-i\partial_\mu A_\mu+A_\mu A_\mu+{}^1/_4 i[\gamma_\mu,\gamma_\nu]F_{\mu\nu}(A),$$

$$\Phi^{(D)}_{(j-D)/2}(\check{\nabla}^2(A);x)=(2\pi)^{-(D+1)}\int d^D\xi\int_\Gamma i\,d\lambda R_{-2-j}(x;\xi,\lambda)\exp(-\lambda), \tag{A.3}$$

$$\delta_{0l}=\sum_{j+k+|\alpha|=l}(\alpha!)^{-1}(\partial_\xi^\alpha\sigma_{2-k})(-i\partial_x)^\alpha R_{-2-j},\quad l=0,1,2,\ldots. \tag{A.4}$$

The contour of integration Γ in (A.3) and (A.8) passes down (from above to below) the imaginary axis, passing round the origin on a small circle to the left; $\alpha=(\alpha_1,\ldots,\alpha_S)$, $|\alpha|=\Sigma\alpha_i$, $\alpha!=\Pi(\alpha_i!)$. R_{-2-j} satisfy the homogeneity relation, and the asymptotic series $\sum_{j=0} R_{-2-j}(x;\xi,\lambda)$ is the symbol of the resolvent $[\check{\nabla}^2(A) - \lambda]$.

From (A.2) and (A.4) the functions R are determined:

$$R_{-2-2l}(x;\xi,\lambda)=(-1)^l(\xi^2-\lambda)^{-(l+1)}({}^1/_4 i[\gamma_\mu,\gamma_\nu]F_{\mu\nu}(A))^l+\ldots; \tag{A.5}$$

$$R_{-2-(2l+1)}(x;\xi,\lambda)=2i(-1)^l(\xi^2-\lambda)^{-(l+2)}\sum_{r=0}^{l}({}^1/_4 i[\gamma_\mu,\gamma_\nu]F_{\mu\nu}(A))^r\xi_\lambda(\partial_\lambda+iA_\lambda(x))({}^1/_4 i[\gamma_\mu,\gamma_\nu]F_{\mu\nu}(A))^{l-r}+\ldots, \tag{A.6}$$

where the ellipses denote terms containing less than 2ℓ of the γ matrices.

Similarly, for $[\check{\nabla}(A)\exp(-\tau\check{\nabla}^2(A))](x,\ x)$ we have

$$[\check{\nabla}(A)\exp(-\tau\check{\nabla}^2(A))](x,x)=\sum_{j=0}^{\infty}\tau^{(j-D-1)/2}\Phi^{(D)}_{(j-D-1)/2}(\check{\nabla}^2(A)\,|\,\check{\nabla}(A))(x), \tag{A.7}$$

$$\Phi^{(D)}_{(j-D-1)/2}(\check{\nabla}^2(A)\,|\,\check{\nabla}(A))(x)=(2\pi)^{-(D+1)}\int d^D\xi\int_\Gamma i\,d\lambda[i\check{\xi}R_{-2-j}(x;\xi,\lambda)+i\check{A}(x)R_{-1-j}(x;\xi,\lambda)]\exp(-\lambda). \tag{A.8}$$

The following expressions were also needed for the calculations:

$$\begin{cases}\mathrm{tr}[\gamma^{(D+1)}\gamma_{\mu_1}\ldots\gamma_{\mu_k})=0 & (k<D).\\ \mathrm{tr}(\gamma^{(D+1)}\gamma_{\mu_1}\ldots\gamma_{\mu_D})=2^{D/2}(-i)^{D(D+1)/2}\varepsilon_{\mu_1\ldots\mu_D}, & D\text{ even},\end{cases} \tag{A.9}$$

$$\begin{cases}\mathrm{tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2k+1}})=0 & (k<{}^1/_2(D-1)),\\ \mathrm{tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_D})=2^{(D-1)/2}(-i)^{D(D+1)/2}\varepsilon_{\mu_1\ldots\mu_D}, & D\text{ odd}.\end{cases} \tag{A.10}$$

Appendix B: Behavior of the Kernel of the Operator $\check{\nabla}^2(A)$ at Large Times

The asymptotic behavior of $(\exp(-\tau\check{\nabla}^2(A)))(x,\ x)$ as $\tau \to \infty$ depends strongly on the behavior of $A_\mu(x)$ as $|x| \to \infty$, and also on D. The kernel is related to the spectral density P by

$$(\exp(-\tau\check{\nabla}^2(A)))(x,x)=\int_{-\infty}^{+\infty}d\lambda P_{\check{\Delta}(A)}(\lambda;x,x)\exp(-\tau\lambda^2)=-\tau^{-1/2}\int_{-\infty}^{+\infty}d\alpha P_{\check{\nabla}(A)}(\alpha\tau^{-1/2};x,x)e^{-\alpha}. \tag{B.1}$$

The asymptotic behavior of this expression as $\tau \to \infty$ is determined by the behavior of P in the limit $\lambda \to 0$. In the limit $\tau \to \infty$ we obtain for (B.1)

$$\Pi_0^{\check{\nabla}(A)},\ \text{if}\ \check{\nabla}(A)\ \text{has zero modes}, \tag{B.2a}$$

$$(\pi/\tau)^{1/2}P_{\check{\nabla}(A)}(0;x,x),\ \text{if}\ \check{\nabla}(A)\ \text{does not have zero modes}. \tag{B.2b}$$

Equation (B.2a) is obtained with allowance for the equation

$$P_{\check{\nabla}(A)}(\lambda;\, x,\, x') = \delta(\lambda)\, \Pi_0^{\check{\nabla}(A)}(x,\, x') + \bar{P}_{\check{\nabla}(A)}(\lambda;\, x,\, x'),$$

where $P_{\check{\nabla}(A)}(0;\, x,\, x') = 0$, and $\Pi_0^{\check{\nabla}(A)}$ is the kernel of the operator of the zero modes.

In the case of (B.2b), P satisfies the relation

$$P_{\hat{\nabla}(A)}(0;\, x,\, x) = [U_{\pm} P_{\check{\nabla}(A^{as})}(0)\, U_{\pm}^{*}](x,\, x), \tag{B.3}$$

where $U_{\pm}$ are the wave operators (in the sense of scattering theory) of the pair $H \equiv \check{\nabla}^2(A)$ and $H_0 \equiv \check{\nabla}^2(A^{as})$ (the total and free quantum-mechanical Hamiltonians, respectively):

$$U_{\pm} \equiv U_{\pm}(H, H_0) = s\text{-}\lim_{t\to\pm\infty} [\exp(itH)\exp(-itH_0)], \tag{B.4}$$

and $A_{\mu}^{as}(x)$ denotes the asymptotic form of $A_{\mu}(x)$ as $|x| \to \infty$ continued to the complete space R^D. In accordance with the general theorems of scattering theory [37], existence and completeness of the wave operators (B.4) is guaranteed if

$$A_{\mu}(x) = A_{\mu}^{as}(x) + O(|x|^{-1-\varepsilon}) \text{ as } |x| \to \infty. \tag{B.5}$$

Note that the standard boundary conditions (1.1) have the form (B.5), where $A_{\mu}^{as}(x) \approx -ig(\hat{x})\,\partial_{\mu} g(\hat{x})$ for $|x|\to\infty$ $(\hat{x}\in S_{\infty}^{D-1})$. If g belongs to the trivial homotopy class of the group $\pi_{D-1}U(n))$ (in particular, if $\pi_{D-1}(U(n)) = 0$, which is true for $D - 1 < 2n$), g can be smoothly extended from S_{∞}^{D-1} to the whole of R^D [33].

In our case, the free Hamiltonian has the form

$$H_0 \equiv \check{\nabla}^2(-ig^{-1}\partial g) = g^{-1}(-\partial^2)\, g. \tag{B.6a}$$

If the element g is homotopically nontrivial on S_{∞}^{D-1}, it can be smoothly extended to the region $V_2 = \{x;\ \ |x| \geq R > 0\}$, while within $V_1 = \{x;\ \ |x| < R\}$ there will be singularities. In this case, H_0 can be defined as follows:

$$H_0 \equiv \check{\nabla}^2(A^{as}) \begin{cases} g^{-1}(-\partial^2)\, g & \text{on } V_2, \\ -\partial^2 & \text{on } V_1. \end{cases} \tag{B.6b}$$

In both cases (B.6a) and (B.6b), $(\exp(-\tau\check{\nabla}^2(A^{as}))(x,\, x'))$ can be readily calculated, and, substituting the obtained result in (B.1), we obtain

$$(\exp(-\tau\check{\nabla}^2(A)))\,(x, x') = [U_{\pm}\exp(-\tau\check{\nabla}^2(A^{as}))\, U_{\pm}^{*}](x, x) \underset{\tau\to\infty}{\approx} (4\pi\tau)^{-D/2} w_{\pm}(x)\, w_{\pm}^{*}(x),$$

$$w(x) \equiv \int d^D y\, U_{\pm}(x, y)\, g^{-1}(y) \ \text{ (in the case (B.6a))}, \tag{B.7}$$

$$w(x) = \int d^D y\, U_{\pm}(x, y)\, [\theta(|y| - R)\, g^{-1}(|y|, \hat{y}) + \theta(R - |y|)\, g^{-1}(R, \hat{y})], \quad \hat{y} = \frac{y}{|y|} \ \text{ (in the case (B.6b))}.$$

From Eqs. (B.7), (B.2b), and (B.3) with the boundary conditions (1.1), we obtain

$$P_{\check{\nabla}(A)}(0;\, x,\, x) = \begin{cases} \delta(0)\, \Pi_0^{\check{\nabla}(A)}(x,\, x) = \infty, & \text{if } \check{\nabla}(A) \text{ has zero modes}, \\ [U_{\pm} P_{\check{\nabla}(A^{as})}(0)\, U_{\pm}^{*}](x,\, x), & \text{if } \check{\nabla}(A) \text{ does not have zero modes}. \end{cases} \tag{B.8}$$

Finally, we note that $P_{\check{\nabla}(A)}(0;\, x,\, x)$ may be nonzero and finite provided $F_{\mu\nu}(A)$ does not vanish as $|x| \to \infty$. In particular, if $A_{\mu}(x)$ is static $(A_{\mu}^{as}(x) = A_{\mu}(\mathbf{x}))$:

$$\check{\nabla}(A) \equiv \check{\nabla}_D = i\gamma_0\partial_0 + \mathscr{D}_{D-1},$$

$$\mathscr{D}_{D-1} = \begin{cases} \check{\nabla}_{D-1} + i\gamma_0 A_0(\mathbf{x}), & \text{if } (D-1) \text{ even}, \\ \begin{pmatrix} 0 & \check{\nabla}_{D-1} + iA_0(\mathbf{x}) \\ \check{\nabla}_{D-1} + iA_0(\mathbf{x}) & 0 \end{pmatrix}, & \text{if } (D-1) \text{ odd}, \end{cases} \tag{B.9}$$

$$\check{\nabla}_{D-1} = \gamma_k(\partial_k + A_k(\mathbf{x})), \qquad \mathbf{x} \equiv (x^1, \ldots, x^{D-1}), \qquad k = 1, \ldots, D-1,$$

and using (B.1), we obtain

$$(\exp(-\tau\check{\nabla}_D^2))\,(x, x) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \exp(-\tau\omega^2)\, (\exp(-\tau(2\omega A_0 + \mathscr{D}_{D-1}^2)))\,(x, x) \underset{\tau\to\infty}{\approx} (4\pi\tau)^{-1/2} (\exp(-\tau\mathscr{D}_{D-1}^2))\,(\mathbf{x}, \mathbf{x}). \tag{B.10}$$

From (B.10) and (B.2b) for static field $A_\mu(\mathbf{x})$ and in the presence of zero modes of $\mathscr{D}_{D-1}$ we find

$$
\begin{gathered}
P_{\check{\nabla}_D}(0; x, x) = (2\pi)^{-1}\,\Pi_0^{\mathscr{D}_{D-1}}(\mathbf{x}, \mathbf{x}), \\
\Pi_0^{\mathscr{D}_{D-1}}(\mathbf{x}, \mathbf{x}) = \begin{cases} \Pi_0^{(\check{\nabla}_{D-1}+i\gamma_0 A_0)}(\mathbf{x}, \mathbf{x}), & (D-1)\ \text{even}, \\ \Pi_0^{(\check{\nabla}_{D-1}+iA_0)}(\mathbf{x}, \mathbf{x}) + \Pi_0^{(\check{\nabla}_{D-1}-iA_0)}(\mathbf{x}, \mathbf{x}), & (D-1)\ \text{odd}. \end{cases}
\end{gathered}
\tag{B.11}
$$

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CALCULATION OF QUANTUM CORRECTIONS TO NONTRIVIAL CLASSICAL SOLUTIONS BY MEANS OF THE ZETA FUNCTION

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The calculation of single-loop quantum corrections in field theory by the use of the zeta function is considered. General expressions are obtained for the quantum corrections to soliton masses and the probability of decay of the metastable vacuum in a scalar theory.

1. Introduction

In recent years, various methods have been developed for quantizing nonlinear field theory equations in the neighborhood of classical solutions [1-5]. Nontrivial solutions of such equations are provided, for example, by solitons, walls and bubbles, strings, and monopoles. However, a complete analysis has not yet been made of the radiative corrections to the energy of a monopole and string and to the process of formation of bubbles of metastable vacuum. In principle, even for a theory that is renormalizable in the vacuum sector, the radiative corrections to the nontrivial classical solutions may be divergent, and this would mean that in reality such objects disappear altogether from the physical sector.

There exist various ways for calculating the single-loop quantum corrections to the nontrivial classical solutions of equations of field theory, but they are all in some way related to calculation of functional determinants for specific field-theoretical problems.

A general method for calculating single-loop quantum corrections to localized classical solutions was developed in [3] (see also the literature quoted there), in which the corresponding functional determinant was expressed in terms of the asymptotic behaviors of Jost solutions. Other authors have calculated the single-loop quantum corrections under the restriction to a box space, obtaining therefore discrete eigenvalues, found the product of eigenvalues, and then allowed the dimensions of the box to tend to infinity. Thus, calculations were made of the quantum corrections in (1 + 1) dimensions to the soliton masses in the scalar and supersymmetric $\lambda\varphi^4$ theory [2,6,7] and in the sine-Gordon model [3,7,8], and also to the wall mass in (3 + 1) dimensions in the scalar $\lambda\varphi^4$ theory [9,10]. An elegant method of semiclassical quantization was proposed in [11] for systems of the type of the sine-Gordon model, for which one can obtain a complete set of action-angle variables. Interesting ways of calculating functional determinants were also proposed in [12,13]. There has also been an ever wider use in the calculation of functional determinants of the powerful formal method based on use of the generalized ζ function [14,15] (see also, for example, [16-18]).

The aim of the present paper is to show that the ζ-function method enables one to reproduce readily and in a unified manner a number of well-known results and calculate the single-loop quantum corrections in the scalar theory for some new cases.

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